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TROPICAL FANS AND THE MODULI SPACES OF TROPICAL CURVES

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ABSTRACT. We give a rigorous definition of tropical fans (the "local building blocks for tropical varieties") and their morphisms. For a morphism of tropical fans of the same dimension we show that the number of inverse images (counted with suitable tropical multiplicities) of a point in the target does not depend on the chosen point — a statement that can be viewed as one of the important first steps of tropical intersection theory. As an application we consider the moduli spaces of rational tropical curves (both abstract and in some $\mathbb{R}^r$) together with the evaluation and forgetful morphisms. Using our results this gives new, easy, and unified proofs of various tropical independence statements, e.g. of the fact that the numbers of rational tropical curves (in any $\mathbb{R}^r$) through given points are independent of the points.

1. INTRODUCTION

Tropical geometry has been used recently for many applications in enumerative geometry, both to reprove previously known results (e.g. the Caporaso-Harris formula [GM1] and Kontsevich's formula [GM2] for plane curves) and to find new methods for counting curves (e.g. Mikhalkin's lattice path algorithm [M1]). All these applications involve counting curves with certain conditions, and at some point require an argument why the resulting numbers remain invariant under deformations of the conditions — e.g. why the numbers of tropical plane rational curves of degree d through $3d-1$ points do not depend on the chosen points (when counted with the appropriate tropical multiplicities).

In classical algebraic geometry, the corresponding independence statements simply follow from the fact that the enumerative numbers can be interpreted as intersection numbers of cycles on suitable moduli spaces, and that intersection numbers are always invariant under deformations. Tropically however there is no well-established intersection theory yet, and thus so far it was necessary to find alternative proofs of the independence statements in each case — either by tedious ad-hoc computations (as e.g. in [GM2]) or by relating the tropical numbers to classical ones for which the invariance is known (as e.g. in [M1]).

The goal of our paper is to fill this gap by establishing the very basics of a "tropical intersection theory" up to the point needed so that at least some of the above independence statements follow from general principles of this theory, just as in classical algebraic geometry.

To do so we start in chapter 2 by giving a rigorous definition of the notion of tropical fans (roughly speaking tropical varieties in some vector space all of whose cells are cones with apex at the origin, i.e. a "local picture" of a general tropical variety), morphisms between them, and the image fan of a morphism. Our most important result in this chapter is corollary 2.26 which states that for a morphism of tropical fans of the same dimension, with the target being irreducible, the sum of the multiplicities of the inverse images of a general

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point P in the target is independent of the choice of P. In chapter 3 we establish the moduli spaces of abstract n-marked rational tropical curves as tropical fans, and show that the forgetful maps between them are morphisms of fans. We then use these results in chapter 4 to construct tropical analogues of the spaces of rational stable maps to a toric variety. Again, these spaces will be tropical fans, with the evaluation and forgetful maps being morphisms of fans. In chapter 5 finally, we present two examples of our results by giving new, easy, and generalized proofs of two statements in tropical enumerative geometry that have already occurred earlier in the literature: the statement that the numbers of rational tropical curves in some $\mathbb{R}^r$ of given degree and through fixed affine linear subspaces in general position do not depend on the position of the subspaces, and the statement that the degree of the combined evaluation and forgetful map occurring in the proof of the tropical Kontsevich formula is independent of the choice of the points (see [GM2] proposition 4.4). Whereas earlier tropical proofs of these statements have been very complicated, our new proofs are now just easy applications of our general statement in corollary 2.26.

We believe that our work results not only in a better understanding of the moduli spaces of tropical curves, but also of tropical varieties and their intersection-theoretical properties in general (see [M2]). Subsequent work on a tropical intersection theory building up on the principles of this paper is in progress.

We should mention that a theorem very similar to our corollary 2.26 has recently been proven independently by Bernd Sturmfels and Jenia Tevelev (see [ST] theorem 1.1). In their paper the authors restrict to tropical fans that are tropicalizations of algebraic varieties (note that these objects are just called "tropicalizations" in their work, whereas their term "tropical fan" has a meaning different from ours). As a result, they are able not only to prove the independence statement of our corollary 2.26 but also to show that this invariant is equal to the degree of the corresponding morphism of algebraic varieties. On the other hand, our result is of course applicable in more generality as it does not need the (rather strong) requirement on the tropical fans to be tropicalizations of algebraic varieties. Another consequence of the different point of view in [ST] compared to our work is that the proofs are entirely different: whereas Sturmfels and Tevelev prove their theorem 1.1 by relating the tropical set-up to the algebraic one our proof of corollary 2.26 is entirely combinatorial and does not use any algebraic geometry.

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2. Tropical fans

Throughout this section Λ will denote a finitely generated free abelian group (i.e. a group isomorphic to $\mathbb{Z}^N$ for some $N \geq 0$) and $V := \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ the corresponding real vector space, so that Λ can be considered as a lattice in V. The dual lattice in the dual vector space $V^\vee$ will be denoted $\Lambda^\vee$.

Definition 2.1 (Cones)**.** A *cone* is a subset $\sigma \subset V$ given by finitely many linear integral equalities and (non-strict) inequalities, i.e. a set of the form

$$\sigma = \{x \in V;\ f_i(x) = 0 \text{ for all } i = 1,\dots,n \text{ and } g_j(x) \geq 0 \text{ for all } j = 1,\dots,m\} \qquad (*)$$

for some $f_1,\dots,f_n,g_1,\dots,g_m \in \Lambda^\vee \subset V^\vee$. We denote by $V_\sigma \subset V$ the smallest vector subspace of V that contains σ, and by $\Lambda_\sigma := V_\sigma \cap \Lambda \subset \Lambda$ the smallest sublattice of Λ that contains $\sigma \cap \Lambda$. The *dimension* of σ is defined to be $\dim \sigma := \dim V_\sigma$.

Let σ be a cone in V. A cone $\tau \subset \sigma$ that can be obtained from σ by changing some (maybe none) of the inequalities $g_j(x) \geq 0$ in $(*)$ to equalities $g_j(x) = 0$ is called a *face* of σ. We write this as $\tau \leq \sigma$ (or as $\tau < \sigma$ if in addition $\tau \subsetneq \sigma$). We obviously have $V_\tau \subset V_\sigma$ and $\Lambda_\tau \subset \Lambda_\sigma$ in this case.

Remark and Definition 2.2. It is well-known that a cone can be described equivalently as a subset of V that can be written as

$$\sigma = \{\lambda_1 u_1 + \cdots + \lambda_n u_n;\ \lambda_1,\dots,\lambda_n \in \mathbb{R}_{\geq 0}\}$$

for some integral vectors $u_1,\dots,u_n \in \Lambda$. In this case we say that the cone σ is *generated* by $u_1,\dots,u_n$. A cone is called an *edge* if it can be generated by one vector, and *simplicial* if it can be generated by $\dim \sigma$ vectors (that are then necessarily linearly independent).

If σ is generated by $u_1,\dots,u_n \in \Lambda$ then each face of σ can be generated by a suitable subset of $\{u_1,\dots,u_n\}$. In fact, if σ is simplicial then there is a 1:1 correspondence between faces of σ and subsets of $\{u_1,\dots,u_n\}$. In particular, such a simplicial cone has among its faces exactly n edges, namely $\mathbb{R}_{\geq 0} \cdot u_1,\dots,\mathbb{R}_{\geq 0} \cdot u_n$.

Construction 2.3 (Normal vectors). Let $\tau < \sigma$ be cones in V with $\dim \tau = \dim \sigma - 1$. By definition there is a linear form $g \in \Lambda^\vee$ that is zero on τ, non-negative on σ, and not identically zero on σ. Then g induces an isomorphism $V_\sigma/V_\tau \to \mathbb{R}$ that is non-negative and not identically zero on σ/V_τ, showing that the cone σ/V_τ lies in a unique half-space of $V_\sigma/V_\tau \cong \mathbb{R}$. As moreover $\Lambda_\sigma/\Lambda_\tau \subset V_\sigma/V_\tau$ is isomorphic to $\mathbb{Z}$ there is a unique generator of $\Lambda_\sigma/\Lambda_\tau$ lying in this half-space. We denote it by $u_{\sigma/\tau} \in \Lambda_\sigma/\Lambda_\tau$ and call it the *(primitive) normal vector* of σ relative to τ.

Definition 2.4 (Fans)**.** A *fan* X in V is a finite set of cones in V such that

(a) all faces of the cones in X are also in X; and

(b) the intersection of any two cones in X is a face of each (and hence by (a) also in X).

The set of all k-dimensional cones of X will be denoted $X^{(k)}$. The biggest dimension of a cone in X is called the *dimension* $\dim X$ of X; we say that X is *pure-dimensional* if each inclusion-maximal cone in X has this dimension. A fan is called *simplicial* if all of its cones are simplicial. The union of all cones in X will be denoted $|X| \subset V$.

Example 2.5.

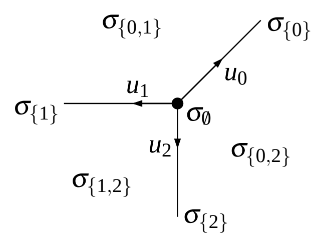
(a) In contrast to the fans considered in toric geometry we do not require our cones to be *strictly* convex, i.e. a cone might contain a straight line through the origin. So e.g. $\{V\}$ is a fan in our sense with only one cone (that has no faces except itself). By abuse of notation we will denote it simply by V.

(b) For $f \in \Lambda^\vee \backslash \{0\}$ the three cones

$$\{x \in V;\ f(x) = 0\}, \quad \{x \in V;\ f(x) \geq 0\}, \quad \{x \in V;\ f(x) \leq 0\}$$

form a fan. We call it the *half-space fan* H_f.

(c) Let $\Lambda = \mathbb{Z}^n$ (and thus $V = \mathbb{R}^n$), let $u_1, \dots, u_n$ be a basis of Λ, and set $u_0 := -u_1 - \dots - u_n$. For each subset $I \subsetneq \{0, \dots, n\}$ we denote by σ_I the simplicial cone of dimension $|I|$ in V spanned by the vectors u_i for $i \in I$. Now fix an integer $k \in \{0, \dots, n\}$ and let L^n_k be the set of all cones σ_I for $|I| \le k$. Then the conditions of definition 2.4 are satisfied since the faces of σ_I are precisely the σ_J with $J \subset I$ by remark 2.2, and moreover we have $\sigma_{I_1} \cap \sigma_{I_2} = \sigma_{I_1 \cap I_2}$ for all $I_1, I_2 \subsetneq \{0, \dots, n\}$. Hence L^n_k is a fan. It is clearly of pure dimension k. For $n = 2$ we obtain the following picture:



where L^2_0 is just the origin, L^2_1 is the "tropical line" given by the origin together with the three edges spanned by u_0, u_1, u_2, and $|L^2_2| = \mathbb{R}^2$.

(d) If X and X' are two fans in $V = \Lambda \otimes \mathbb{R}$ and $V' = \Lambda' \otimes \mathbb{R}$, respectively, it is checked immediately that their product $\{\sigma \times \sigma';\ \sigma \in X,\ \sigma' \in X'\}$ is a fan in $V \times V'$. We call it the *product* of the two fans and denote it by $X \times X'$. We obviously have $|X \times X'| = |X| \times |X'|$.

(e) In the same way, if X and X' are two fans in V it is checked immediately that their intersection $\{\sigma \cap \sigma';\ \sigma \in X,\ \sigma' \in X'\}$ is also a fan in V. We call it the *intersection* of the two fans and denote it by $X \cap X'$. It is clear that $|X \cap X'| = |X| \cap |X'|$.

Definition 2.6 (Subfans)**.** Let X be a fan in V. A fan Y in V is called a *subfan* of X (denoted $Y \subset X$) if each cone of Y is contained in a cone of X. In this case we denote by $C_{Y,X} : Y \to X$ the map that sends a cone $\sigma \in Y$ to the (unique) inclusion-minimal cone of X that contains σ. Note that for a subfan $Y \subset X$ we obviously have $|Y| \subset |X|$ and $\dim C_{Y,X}(\sigma) \ge \dim \sigma$ for all $\sigma \in Y$.

Example 2.7. The intersection of two fans as in example 2.5 (e) is clearly a subfan of both.

Definition 2.8 (Weighted and tropical fans)**.** A *weighted fan* in V is a pair (X, ω_X) where X is a fan of some pure dimension n in V, and $\omega_X : X^{(n)} \to \mathbb{Z}_{>0}$ is a map. We call $\omega_X(\sigma)$ the *weight* of the cone $\sigma \in X^{(n)}$ and write it simply as $\omega(\sigma)$ if no confusion can result. Also, by abuse of notation we will write a weighted fan (X, ω_X) simply as X if the weight function ω_X is clear from the context.

A *tropical fan* in V is a weighted fan (X, ω_X) in V such that for all $\tau \in X^{(\dim X - 1)}$ the *balancing condition*

$$\sum_{\sigma > \tau} \omega_X(\sigma) \cdot u_{\sigma/\tau} = 0 \quad \in V/V_\tau$$

holds (where $u_{\sigma/\tau}$ denotes the primitive normal vector of construction 2.3).

Example 2.9.

(a) Of course, the fan V of example 2.5 (a) with the weight function $\omega(V) := 1$ is a tropical fan (the balancing condition being trivial in this case). The half-space fans

H_f of example 2.5 (b) are tropical fans as well if we define the weight of both maximal cones to be 1: the balancing condition around the cone $\{x \in V;\ f(x) = 0\}$ holds because the primitive normal vectors of the two adjacent maximal cones are negatives of each other. We will see in example 2.15 that the fans L^n_k of example 2.5 (c) are also tropical fans if the weights of all maximal cones are defined to be 1.

(b) Let (X, ω_X) be a weighted fan of dimension n, and let $\lambda \in \mathbb{Q}_{>0}$ such that $\lambda \omega_X(\sigma) \in \mathbb{Z}_{>0}$ for all $\sigma \in X^{(n)}$. Then $(X, \lambda \cdot \omega_X)$ is a weighted fan as well; we will denote it by $\lambda \cdot X$. It is obvious that $\lambda \cdot X$ is tropical if and only if X is.

(c) Let (X, ω_X) and $(X', \omega_{X'})$ be two weighted fans of dimensions n and n' in $V = \Lambda \otimes \mathbb{R}$ and $V' = \Lambda' \otimes \mathbb{R}$, respectively. We consider their product $X \times X'$ of example 2.5 (d) to be a weighted fan by setting $\omega_{X \times X'}(\sigma \times \sigma') := \omega_X(\sigma) \cdot \omega_{X'}(\sigma')$ for $\sigma \in X^{(n)}$ and $\sigma' \in X'^{(n')}$. If moreover (X, ω_X) and $(X', \omega_{X'})$ are tropical then so is $X \times X'$: the cones in $(X \times X')^{(n+n'-1)}$ are of the form $\sigma \times \sigma'$ for $\sigma \in X^{(n-1)}$, $\sigma' \in X'^{(n')}$ or $\sigma \in X^{(n)}$, $\sigma' \in X'^{(n'-1)}$, and the balancing conditions in these cases are easily seen to be induced from the ones of X and X', respectively.

Definition 2.10 (Refinements)**.** Let (X, ω_X) and (Y, ω_Y) be weighted fans in V. We say that (Y, ω_Y) is a *refinement* of (X, ω_X) if

- $Y \subset X$;
- $|Y| = |X|$ (so in particular $\dim Y = \dim X$); and
- $\omega_Y(\sigma) = \omega_X(C_{Y,X}(\sigma))$ for all maximal cones $\sigma \in Y$ (where $C_{Y,X}$ is as in definition 2.6).

We say that the two weighted fans (X, ω_X) and (Y, ω_Y) are *equivalent* (written $(X, \omega_X) \cong (Y, \omega_Y)$) if they have a common refinement.

Example 2.11.

(a) The half-space fans H_f of example 2.5 (b) are all refinements of the trivial fan V of example 2.5 (a) (with the weights all 1 as in example 2.9 (a)).

(b) Let (X, ω_X) be a weighted fan in V, and let Y be any fan in V with $|Y| \supset |X|$. Then the intersection $X \cap Y$ (see examples 2.5 (e) and 2.7) is a refinement of (X, ω_X) by setting $\omega_{X \cap Y}(\sigma) := \omega_X(C_{X \cap Y, X}(\sigma))$ for all maximal cones $\sigma \in X \cap Y$. In fact, this construction will be our main source for refinements in this paper.

Note that in the special case when (X, ω_X) and (Y, ω_Y) are both weighted fans and $|Y| = |X|$ we can form both intersections $X \cap Y$ and $Y \cap X$. The underlying non-weighted fans of these two intersections are of course always the same, but the weights may differ as they are by construction always induced by the first fan.

(c) The equivalence of weighted fans is in fact an equivalence relation: if $X_1 \cong X_2$ with common refinement Y_1, and $X_2 \cong X_3$ with common refinement Y_2, then $Y_1 \cap Y_2$ as in (b) is a common refinement of X_1 and X_3 (note that $Y_1 \cap Y_2 = Y_2 \cap Y_1$ in this case as both Y_1 and Y_2 are refinements of X_2).

(d) Let (Y, ω_Y) be a refinement of a weighted fan (X, ω_X) of dimension n. We claim that (Y, ω_Y) is tropical if and only if (X, ω_X) is. In fact, for $\tau \in Y^{(n-1)}$ we have the following two cases:

- $\dim C_{Y,X}(\tau) = n$: Then the n-dimensional cones in Y that have τ as a face must be contained in $C_{Y,X}(\tau)$. It follows that there are exactly two of them, with opposite normal vectors and the same weight (equal to $\omega_X(C_{Y,X}(\tau))$). In particular, the balancing condition for Y always holds at such cones.
- $\dim C_{Y,X}(\tau) = n-1$: Then the map $\sigma \mapsto C_{Y,X}(\sigma)$ gives a 1:1 correspondence between cones $\sigma \in Y^{(n)}$ with $\sigma > \tau$ and cones $\sigma' \in X^{(n)}$ with $\sigma' > C_{Y,X}(\tau)$. As this correspondence preserves weights and normal vectors the balancing condition for Y at τ is equivalent to that for X at $C_{Y,X}(\tau)$. As each $(n-1)$-dimensional cone of X occurs as $C_{Y,X}(\tau)$ for some $\tau \in Y^{(n-1)}$ this means that the balancing conditions for X and Y are equivalent.

Consequently, the notion of weighted fans being tropical is well-defined on equivalence classes.

Definition 2.12 (Marked fans)**.** A *marked fan* in V is a pure-dimensional simplicial fan X in V together with the data of vectors $v_\sigma \in (\sigma \backslash \{0\}) \cap \Lambda$ for all $\sigma \in X^{(1)}$ (i.e. v_σ is an integral vector generating the edge σ).

Construction 2.13*.* Let X be a marked fan of dimension n.

(a) Let $\sigma \in X^{(k)}$ be a k-dimensional cone in X. As σ is simplicial by assumption we know by remark 2.2 that there are exactly k edges $\sigma_1, \dots, \sigma_k \in X^{(1)}$ that are faces of σ, and that the vectors $v_{\sigma_1}, \dots, v_{\sigma_k}$ generating these edges are linearly independent. Hence

$$\tilde{\Lambda}_\sigma := \mathbb{Z} v_{\sigma_1} + \cdots + \mathbb{Z} v_{\sigma_k}$$

is a sublattice of Λ_σ of full rank, and consequently $\Lambda_\sigma / \tilde{\Lambda}_\sigma$ is a finite abelian group. We set $\omega(\sigma) := |\Lambda_\sigma / \tilde{\Lambda}_\sigma| \in \mathbb{Z}_{>0}$. In particular, this makes the marked fan X into a weighted fan. In this paper marked fans will always be considered to be weighted fans in this way.

(b) Let $\sigma \in X^{(k)}$ and $\tau \in X^{(k-1)}$ with $\sigma > \tau$. As in (a) there are then exactly $k-1$ edges in $X^{(1)}$ that are faces of τ, and k edges that are faces of σ. There is therefore exactly one edge $\sigma' \in X^{(1)}$ that is a face of σ but not of τ. The corresponding vector $v_{\sigma'}$ will be denoted $v_{\sigma/\tau} \in \Lambda$; it can obviously also be thought of as a "normal vector" of σ relative to τ. Note however that in contrast to the normal vector $u_{\sigma/\tau}$ of construction 2.3 it is defined in Λ and not just in Λ/Λ_τ, and that it need not be primitive.

Lemma 2.14. *Let X be a marked fan of dimension n (and hence a weighted fan by construction 2.13 (a)). Then X is a tropical fan if and only if for all $\tau \in X^{(n-1)}$ the balancing condition*

$$\sum_{\sigma > \tau} v_{\sigma/\tau} = 0 \quad \in V/V_\tau$$

holds (where the vectors $v_{\sigma/\tau}$ are as in construction 2.13 (b)).

Proof. We have to show that the given equations coincide with the balancing condition of definition 2.8. For this it obviously suffices to prove that

$$\omega(\sigma) \cdot u_{\sigma/\tau} = \omega(\tau) \cdot v_{\sigma/\tau} \quad \in \Lambda_\sigma/\Lambda_\tau \qquad \text{(and hence in } V_\sigma/V_\tau)$$

for all $\sigma \in X^{(n)}$ and $\tau \in X^{(n-1)}$ with $\sigma > \tau$. Using the isomorphism $\Lambda_\sigma/\Lambda_\tau \cong \mathbb{Z}$ of construction 2.3 this is equivalent to the equation

$$\omega(\sigma) = \omega(\tau) \cdot |\Lambda_\sigma/(\Lambda_\tau + \mathbb{Z} v_{\sigma/\tau})|$$

in $\mathbb{Z}$, i.e. using construction 2.13 (a) and the relation $\tilde{\Lambda}_\sigma = \tilde{\Lambda}_\tau + \mathbb{Z} v_{\sigma/\tau}$ to the equation

$$|\Lambda_\sigma/(\tilde{\Lambda}_\tau + \mathbb{Z} v_{\sigma/\tau})| = |\Lambda_\tau/\tilde{\Lambda}_\tau| \cdot |\Lambda_\sigma/(\Lambda_\tau + \mathbb{Z} v_{\sigma/\tau})|.$$

But this follows immediately from the exact sequence

$$0 \longrightarrow \Lambda_\tau/\tilde{\Lambda}_\tau \longrightarrow \Lambda_\sigma/(\tilde{\Lambda}_\tau + \mathbb{Z} v_{\sigma/\tau}) \longrightarrow \Lambda_\sigma/(\Lambda_\tau + \mathbb{Z} v_{\sigma/\tau}) \longrightarrow 0.$$

□

Example 2.15. We consider again the fans L_k^n of example 2.5 (c) and make them into marked fans by setting $v_{\sigma_{\{i\}}} := u_i$ for $i = 0, \dots, n$. We claim that L_k^n becomes a tropical fan in this way. In fact, to prove this we just have to check the balancing condition of lemma 2.14. By symmetry it suffices to do this at the $(k-1)$-dimensional cone $\sigma_{\{0,\dots,k-2\}}$. The k-dimensional cones in L_k^n containing this cone are exactly $\sigma_{\{0,\dots,k-2,i\}}$ for $i = k-1, \dots, n$. Consequently, the balancing condition that we have to check is just

$$u_{k-1} + \cdots + u_n = 0 \quad \in \mathbb{R}^n/(\mathbb{R} u_0 + \cdots + \mathbb{R} u_{k-2}),$$

which is obvious since $u_0 + \cdots + u_n = 0$. So the L_k^n are tropical fans of dimension k in $\mathbb{R}^n$. We can think of them as the "standard k-dimensional linear subspace in $\mathbb{R}^n$".

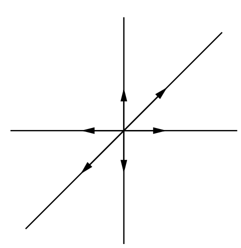
Definition 2.16 (Irreducible fans)**.** Let X be a tropical fan in V. We say that X is *irreducible* if there is no tropical fan Y of the same dimension in V with $|Y| \subsetneq |X|$.

Remark 2.17. Note that the condition of irreducibility remains unchanged under refinements, i.e. it is well-defined on equivalence classes of tropical fans.

Example 2.18. Tropical lines, i.e. the tropical fans L_1^n of example 2.5 (c) (see also example 2.15) are irreducible: any 1-dimensional fan Y in $\mathbb{R}^n$ with $|Y| \subsetneq |L_1^n|$ would have to be obtained by simply removing some of the edges of L_1^n (and possibly changing the weights at the remaining ones), and it is clear that doing so would spoil the balancing condition at the origin since the remaining edge vectors are linearly independent. (In fact, all linear spaces L_k^n of example 2.5 (c) are irreducible, but we will not prove this here as we will not need this result.)

In the same way it follows of course that $\mathbb{R}^1$ (and in fact any tropical fan that is just a straight line through the origin in some $\mathbb{R}^n$) is irreducible.

Remark 2.19. Although there is a notion of irreducible tropical fans it should be noted that there is nothing like a unique decomposition into irreducible components. The easiest example for this is probably the following 1-dimensional tropical fan in $\mathbb{R}^2$ (where the vectors in the picture denote the markings as in definition 2.12).



By example 2.18 it can be realized as a union of irreducible tropical fans either as a union of three straight lines through the origin, or as a union of the tropical line L_1^2 of example 2.5 (c) and the same line mirrored at the origin.

Proposition 2.20. *The product of two irreducible tropical fans (see example 2.9 (c)) is irreducible.*

Proof. Let X and X' be irreducible tropical fans of dimensions n and n' in $V = \Lambda \otimes \mathbb{R}$ and $V' = \Lambda' \otimes \mathbb{R}$, respectively, and let Y be a tropical fan of dimension $n+n'$ in $V \times V'$ with $|Y| \subsetneq |X \times X'|$. By passing from Y to the refinement $Y \cap (X \times X')$ (see example 2.11 (b) and (d)) we may assume that $Y \subset X \times X'$. For any $\sigma \times \sigma' \in (X \times X')^{(n+n')}$ we claim that $A(\sigma \times \sigma') := C^{-1}_{Y,X\times X'}(\sigma \times \sigma') \cap Y^{(n+n')}$ is either empty or consists of cones that cover $\sigma \times \sigma'$ and all have the same weight. In fact, if this was not the case then there would have to be a cone $\tau \in Y^{(n+n'-1)}$ with $C_{Y,X\times X'}(\tau) = \sigma \times \sigma'$ so that to the two sides of τ in $\sigma \times \sigma'$ there is either only one cone of $A(\sigma \times \sigma')$ or two cones with different weights — and in both cases the balancing condition for Y would be violated at τ.

Let us now define a purely $(n+n')$-dimensional fan Z in $V \times V'$ by taking all cones $\sigma \times \sigma' \in (X \times X')^{(n+n')}$ for which $A(\sigma \times \sigma') \neq \emptyset$, together with all faces of these cones. Associating to such a cone $\sigma \times \sigma' \in Z^{(n+n')}$ the weight of the cones in $A(\sigma \times \sigma')$ we make Z into a tropical fan of dimension $n+n'$ — in fact it is just a tropical fan of which Y is a refinement.

As $\dim Z = n+n'$ and $|Z| = |Y| \subsetneq |X \times X'|$ there must be cones $\sigma_1 \times \sigma_1' \in Z$ and $\sigma_0 \times \sigma_0' \in (X \times X') \backslash Z$. We distinguish two cases:

- $\sigma_0 \times \sigma_1' \in (X \times X') \backslash Z$: Then we construct a purely n-dimensional fan X_0 in V by taking all cones $\sigma \in X^{(n)}$ with $\sigma \times \sigma_1' \in Z$, together with all faces of these cones. Setting $\omega_{X_0}(\sigma) := \omega_Z(\sigma \times \sigma_1')$ the fan X_0 becomes an n-dimensional tropical fan, with the balancing condition inherited from Z (in fact, the balancing condition around a cone $\tau \in X_0^{(n-1)}$ follows from the one around $\tau \times \sigma_1'$ in Z). As $|X_0|$ is neither empty (since $\sigma_1 \in X_0$) nor all of $|X|$ (since $\sigma_0 \notin X_0$) this is a contradiction to X being irreducible.
- $\sigma_0 \times \sigma_1' \in Z$: This case follows in the same way by considering the purely n'-dimensional fan Y_0 in V' given by all cones $\sigma' \in Y^{(n')}$ with $\sigma_0 \times \sigma' \in Z$, together with all faces of these cones — leading to a contradiction to X' being irreducible.

□

Lemma 2.21. *Let X and Y be tropical fans of dimension n in V. Assume that $|Y| \subset |X|$, and that X is irreducible. Then $Y \cong \lambda \cdot X$ for some $\lambda \in \mathbb{Q}_{>0}$ (see example 2.9 (b)).*

Proof. As X is irreducible we have $|Y| = |X|$. Replacing X and Y by the refinements $X \cap Y$ and $Y \cap X$ respectively (see example 2.11 (b)) we may assume that X and Y consist of the same cones (with possibly different weights). Let $\lambda := \min_{\sigma \in X^{(n)}} \omega_Y(\sigma)/\omega_X(\sigma) > 0$, and let $\alpha \in \mathbb{Z}_{>0}$ with $\alpha\lambda \in \mathbb{Z}$. Consider the new weight function $\omega(\sigma) = \alpha(\omega_Y(\sigma) - \lambda\, \omega_X(\sigma))$ for $\sigma \in X^{(n)}$. By construction it takes values in $\mathbb{Z}_{\geq 0}$, with value 0 occurring at least once. Construct a new weighted fan Z from this weight function by taking all cones $\sigma \in X^{(n)}$ with $\omega(\sigma) > 0$, together with all faces of these cones. Then (Z, ω) is a tropical fan as the balancing condition is linear in the weights. It does not cover $|X|$ since at least one maximal cone has been deleted from X by construction. Hence it must be empty as X was assumed to be irreducible. This means that $\omega_Y(\sigma) - \lambda\, \omega_X(\sigma) = 0$ for all $\sigma \in X^{(n)}$. □

Definition 2.22 (Morphisms of fans)**.** Let X be a fan in $V = \Lambda \otimes \mathbb{R}$, and let Y be a fan in $V' = \Lambda' \otimes \mathbb{R}$. A *morphism* $f : X \to Y$ is a map f from $|X| \subset V$ to $|Y| \subset V'$ such that f is $\mathbb{Z}$-linear, i.e. induced by a linear map from Λ to Λ'. By abuse of notation we will often denote the corresponding linear maps from Λ to Λ' and from V to V' by the same letter f (note however that these latter maps are not determined uniquely by $f : X \to Y$ if $|X|$ does not span V). A morphism of weighted fans is a morphism of fans (i.e. there are no conditions on the weights).

Remark 2.23. It is obvious from the definition that morphisms from (X, ω_X) to (Y, ω_Y) are in one-to-one correspondence with morphisms from any refinement of (X, ω_X) to any refinement of (Y, ω_Y) simply by using the same map $f : |X| \to |Y|$.

Construction 2.24. Let X be a purely n-dimensional fan in $V = \Lambda \otimes \mathbb{R}$, and let Y be any fan in $\Lambda' \otimes \mathbb{R}$. For any morphism $f : X \to Y$ we will construct an *image fan* $f(X)$ in V' (which is empty or of pure dimension n) as follows. Consider the collection of cones

$$Z := \{ f(\sigma);\ \sigma \in X \text{ contained in a maximal cone of } X \text{ on which } f \text{ is injective}\}$$

in V'. Note that Z is in general not a fan in V': it satisfies condition (a) of definition 2.4, but in general not (b) (since e.g. the images of some maximal cones might overlap in a region of dimension n). To make it into one choose linear forms $g_1, \dots, g_N \in \Lambda'^{\vee}$ such that each cone $f(\sigma) \in Z$ can be written as

$$f(\sigma) = \{x \in V';\ g_i(x) = 0,\ g_j(x) \geq 0,\ g_k(x) \leq 0 \text{ for all } i \in I_\sigma, j \in J_\sigma, k \in K_\sigma\} \qquad (*)$$

for suitable index sets $I_\sigma, J_\sigma, K_\sigma \in \{1, \dots, N\}$. Now we replace X by the fan $\tilde{X} = X \cap H_{g_1 \circ f} \cap \cdots \cap H_{g_N \circ f}$ (see example 2.5 (b)). This fan satisfies $|\tilde{X}| = |X|$, and by definition each cone of $\tilde{X}$ is of the form

$$\{x \in V;\ x \in \sigma,\ g_i(f(x)) = 0,\ g_j(f(x)) \geq 0,\ g_k(f(x)) \leq 0 \text{ for all } i \in I, j \in J, k \in K\}$$

for some $\sigma \in X$ and a partition $I \cup J \cup K = \{1, \dots, N\}$. By $(*)$ the image of such a cone under f is of the form

$$\begin{aligned}
&\{x \in V';\ x \in f(\sigma),\ g_i(x) = 0,\ g_j(x) \geq 0,\ g_k(x) \leq 0 \text{ for all } i \in I, j \in J, k \in K\} \\
&\quad = \{x \in V';\ g_i(x) = 0,\ g_j(x) \geq 0,\ g_k(x) \leq 0 \text{ for all } i \in I', j \in J', k \in K'\}
\end{aligned}$$

for some partition $I' \cup J' \cup K' = \{1, \dots, N\}$, i.e. it is a cone of the fan $H_{g_1} \cap \cdots \cap H_{g_N}$. Hence the set of cones

$$\tilde{Z} := \{ f(\sigma);\ \sigma \in \tilde{X} \text{ contained in a maximal cone of } \tilde{X} \text{ on which } f \text{ is injective}\}$$

consists of cones in the fan $H_{g_1} \cap \cdots \cap H_{g_N}$. As these cones now satisfy condition (b) of definition 2.4 it follows that $\tilde{Z}$ is a fan in V'. It is obvious that it is of pure dimension n.

If moreover X is a weighted fan then $\tilde{X}$ will be a weighted fan as well (see example 2.11 (b)). In this case we make $\tilde{Z}$ into a weighted fan by setting

$$\omega_{\tilde{Z}}(\sigma') := \sum_{\sigma \in \tilde{X}^{(n)} : f(\sigma) = \sigma'} \omega_{\tilde{X}}(\sigma) \cdot |\Lambda'_{\sigma'} / f(\Lambda_\sigma)|$$

for all $\sigma' \in \tilde{Z}^{(n)}$ (see definition 2.1; note that $f(\Lambda_\sigma)$ is a sublattice of $\Lambda'_{\sigma'}$ of full rank).

We denote the fan $(\tilde{Z}, \omega_{\tilde{Z}})$ by $f(X)$ and call it the *image fan* of f. It is clear from the construction that the equivalence class of $f(X)$ remains unchanged if we replace the $g_1, \dots, g_N$ by any larger set of linear forms (this would just lead to a refinement), and hence also if we replace them by any other set of linear forms satisfying $(*)$. Hence the equivalence class

of $f(X)$ does not depend on any choices we made. It is obvious that the equivalence class of $f(X)$ depends in fact only on the equivalence class of X.

By abuse of notation we will usually drop the tilde from the above notation and summarize the construction above as follows: given a weighted fan X of dimension n, an arbitrary fan Y, and a morphism $f : X \to Y$, we may assume after passing to an equivalent fan for X that

$$f(X) = \{ f(\sigma);\ \sigma \in X \text{ contained in a maximal cone of } X \text{ on which } f \text{ is injective}\}$$

is a fan. We consider this to be a weighted fan of dimension n by setting

$$\omega_{f(X)}(\sigma') := \sum_{\sigma \in X^{(n)} : f(\sigma)=\sigma'} \omega_X(\sigma) \cdot |\Lambda'_{\sigma'}/f(\Lambda_\sigma)|$$

for $\sigma' \in f(X)^{(n)}$. This weighted fan is well-defined up to equivalence.

Proposition 2.25. *Let X be an n-dimensional tropical fan in $V = \Lambda \otimes \mathbb{R}$, Y an arbitrary fan in $V' = \Lambda' \otimes \mathbb{R}$, and $f : X \to Y$ a morphism. Then $f(X)$ is an n-dimensional tropical fan as well (if it is not empty).*

Proof. Since we have already seen in construction 2.24 that $f(X)$ is a weighted fan of dimension n we just have to check the balancing condition of definition 2.8. As in this construction we may assume that $f(X)$ just consists of the cones $f(\sigma)$ for all $\sigma \in X$ contained in a maximal cone on which f is injective. Let $\tau' \in f(X)^{(n-1)}$, and let $\tau \in X^{(n-1)}$ with $f(\tau) = \tau'$. Applying f to the balancing condition for X at τ we get

$$\sum_{\sigma > \tau} \omega_X(\sigma) \cdot f(u_{\sigma/\tau}) = 0 \quad \in V'/V'_{\tau'}$$

for all $\tau \in X^{(n-1)}$. Now let $\sigma' \in f(X)$ be a cone with $\sigma' > \tau'$, and $\sigma \in X$ with $\sigma > \tau$ such that $f(\sigma) = \sigma'$. Note that the primitive normal vector $u_{\sigma'/\tau'}$ is related to the (possibly non-primitive) vector $f(u_{\sigma/\tau})$ by

$$f(u_{\sigma/\tau}) = |\Lambda'_{\sigma'}/(\Lambda'_{\tau'} + \mathbb{Z}\, f(u_{\sigma/\tau}))| \cdot u_{\sigma'/\tau'}$$

if f is injective on σ, and $f(u_{\sigma/\tau}) = 0$ otherwise. Inserting this into the above balancing condition, and using the exact sequence

$$0 \longrightarrow \Lambda'_{\tau'}/f(\Lambda_\tau) \longrightarrow \Lambda'_{\sigma'}/f(\Lambda_\sigma) \longrightarrow \Lambda'_{\sigma'}/(\Lambda'_{\tau'} + \mathbb{Z}\, f(u_{\sigma/\tau})) \longrightarrow 0$$

we conclude that

$$\sum_{\sigma > \tau} \omega_X(\sigma) \cdot |\Lambda'_{\sigma'}/f(\Lambda_\sigma)| \cdot u_{\sigma'/\tau'} = 0 \quad \in V'/V'_{\tau'},$$

where the sum is understood to be taken over only those $\sigma > \tau$ on which f is injective.

Let us now sum these equations up for all τ with $f(\tau) = \tau'$. The above sum then simply becomes a sum over all σ with $f(\sigma) > \tau'$ (note that each such σ occurs in the sum exactly once since f is injective on σ so that σ cannot have two distinct codimension-1 faces that both map to τ'). Splitting this sum up according to the cone $f(\sigma)$ we get

$$\sum_{\sigma' > \tau'} \ \sum_{\sigma : f(\sigma)=\sigma'} \omega_X(\sigma) \cdot |\Lambda'_{\sigma'}/f(\Lambda_\sigma)| \cdot u_{\sigma'/\tau'} = 0 \quad \in V'/V'_{\tau'}.$$

But using the definition of the weights of $f(X)$ of construction 2.24 this is now simply the balancing condition

$$\sum_{\sigma' > \tau'} \omega_{f(X)}(\sigma') \cdot u_{\sigma'/\tau'} = 0 \quad \in V'/V'_{\tau'}$$

for $f(X)$. □

Corollary 2.26. *Let X and Y be tropical fans of the same dimension n in $V = \Lambda \otimes \mathbb{R}$ and $V' = \Lambda' \otimes \mathbb{R}$, respectively, and let $f : X \to Y$ be a morphism. Assume that Y is irreducible. Then there is a fan Y_0 in V' of smaller dimension with $|Y_0| \subset |Y|$ such that*

(a) *each point $Q \in |Y| \backslash |Y_0|$ lies in the interior of a cone $\sigma'_Q \in Y$ of dimension n;*

(b) *each point $P \in f^{-1}(|Y| \backslash |Y_0|)$ lies in the interior of a cone $\sigma_P \in X$ of dimension n;*

(c) *for $Q \in |Y| \backslash |Y_0|$ the sum*

$$\sum_{P \in |X| : f(P) = Q} \mathrm{mult}_P\, f$$

does not depend on Q, where the multiplicity $\mathrm{mult}_P\, f$ *of f at P is defined to be*

$$\mathrm{mult}_P\, f := \frac{\omega_X(\sigma_P)}{\omega_Y(\sigma'_Q)} \cdot |\Lambda'_{\sigma'_Q} / f(\Lambda_{\sigma_P})|$$

Proof. Consider the tropical fan $f(X)$ in V' (see construction 2.24 and proposition 2.25). If $f(X) = \emptyset$ (i.e. if there is no maximal cone of X on which f is injective) the statement of the corollary is trivial. Otherwise $f(X)$ has dimension n and satisfies $|f(X)| \subset |Y|$, so as Y is irreducible it follows by lemma 2.21 that $f(X) \cong \lambda \cdot Y$ for some $\lambda \in \mathbb{Q}_{>0}$. After passing to equivalent fans we may assume that $f(X)$ and Y consist of the same cones, and that these are exactly the cones of the form $f(\sigma)$ for $\sigma \in X$ contained in a maximal cone on which f is injective. Now let Y_0 be the fan consisting of all cones of Y dimension less than n. Then (a) and (b) hold by construction. Moreover, each $Q \in |Y| \backslash |Y_0|$ lies in the interior of a unique n-dimensional cone σ', and there is a 1:1 correspondence between points $P \in f^{-1}(Q)$ and n-dimensional cones σ in X with $f(\sigma) = \sigma'$. So we conclude that

$$\sum_{P : f(P) = Q} \mathrm{mult}_P\, f = \sum_{\sigma : f(\sigma) = \sigma'} \frac{\omega_X(\sigma)}{\omega_Y(\sigma')} \cdot |\Lambda'_{\sigma'} / f(\Lambda_\sigma)| = \frac{\omega_{f(X)}(\sigma')}{\omega_Y(\sigma')} = \lambda$$

does not depend on Q. □

Corollary 2.27. *In the situation and with the notation of corollary 2.26 assume moreover that X and Y are marked fans as in definition 2.12, and that their structure of tropical fans has been induced by these data as in construction 2.13 (a). Let $\sigma_1, \dots, \sigma_n \in X^{(1)}$ be the 1-dimensional faces of σ_P, and let $\sigma'_1, \dots, \sigma'_n \in Y^{(1)}$ be the 1-dimensional faces of $\sigma'_Q = \sigma'_{f(P)}$. Then* $\mathrm{mult}_P\, f$ *is equal to the absolute value of the determinant of the matrix for the linear map $f|_{V_{\sigma_P}} : V_{\sigma_P} \to V'_{\sigma'_Q}$ in the bases $\{v_{\sigma_1}, \dots, v_{\sigma_n}\}$ and $\{v_{\sigma'_1}, \dots, v_{\sigma'_n}\}$.*

Proof. It is well-known that $|\Lambda'_{\sigma'_Q} / f(\Lambda_{\sigma_P})|$ is the determinant of the linear map $f|_{V_{\sigma_P}} : V_{\sigma_P} \to V'_{\sigma'_Q}$ with respect to lattice bases of Λ_{σ_P} and $\Lambda'_{\sigma'_Q}$. The statement of the corollary now follows since the base change to $\{v_{\sigma_1}, \dots, v_{\sigma_n}\}$ and $\{v_{\sigma'_1}, \dots, v_{\sigma'_n}\}$ clearly leads to factors in the determinant of $\omega_X(\sigma_P)$ and $1/\omega_Y(\sigma'_Q)$, respectively. □

3. TROPICAL $\mathscr{M}_{0,n}$, GRASSMANNIANS, AND THE SPACE OF TREES

In this chapter we want to show that the moduli space $\mathscr{M}_{0,n,\mathrm{trop}}$ of rational n-marked abstract tropical curves is a tropical fan in the sense of definition 2.8, and that the forgetful map is a morphism of fans.

Let us start by recalling the relevant definitions from [GM2].

Definition 3.1 (Graphs)**.**

(a) Let $I_1,\dots,I_n \subset \mathbb{R}$ be a finite set of closed, bounded or half-bounded real intervals. We pick some (not necessarily distinct) boundary points $P_1,\dots,P_k,Q_1,\dots,Q_k \in I_1 \coprod \cdots \coprod I_n$ of these intervals. The topological space Γ obtained by identifying P_i with Q_i for all $i=1,\dots,k$ in $I_1 \coprod \cdots \coprod I_n$ is called a *graph*. As usual, the *genus* of Γ is simply its first Betti number $\dim H_1(\Gamma,\mathbb{R})$.

(b) For a graph Γ the boundary points of the intervals $I_1,\dots,I_n$ are called the *flags*, their image points in Γ the *vertices* of Γ. If F is such a flag then its image vertex in Γ will be denoted ∂F. For a vertex V the number of flags F with $\partial F = V$ is called the *valence* of V and denoted $\operatorname{val} V$. We denote by Γ^0 and Γ' the sets of vertices and flags of Γ, respectively.

(c) The open intervals $I_1^\circ,\dots,I_n^\circ$ are naturally open subsets of Γ; they are called the *edges* of Γ. An edge will be called bounded (resp. unbounded) if its corresponding open interval is. We denote by Γ^1 (resp. Γ_0^1 and Γ_∞^1) the set of edges (resp. bounded and unbounded edges) of Γ. Every flag $F \in \Gamma'$ belongs to exactly one edge that we will denote by $[F] \in \Gamma^1$. The unbounded edges will also be called the *ends* of Γ.

Definition 3.2 (Abstract tropical curves)**.** A (rational, abstract) tropical curve is a connected graph Γ of genus 0 all of whose vertices have valence at least 3. An *n-marked tropical curve* is a tuple $(\Gamma,x_1,\dots,x_n)$ where Γ is a tropical curve and $x_1,\dots,x_n \in \Gamma_\infty^1$ are distinct unbounded edges of Γ. Two such marked tropical curves $(\Gamma,x_1,\dots,x_n)$ and $(\tilde\Gamma,\tilde x_1,\dots,\tilde x_n)$ are called isomorphic (and will from now on be identified) if there is a homeomorphism $\Gamma \to \tilde\Gamma$ mapping x_i to $\tilde x_i$ for all i and such that every edge of Γ is mapped bijectively onto an edge of $\tilde\Gamma$ by an affine map of slope ± 1, i.e. by a map of the form $t \mapsto a \pm t$ for some $a \in \mathbb{R}$. The space of all n-marked tropical curves (modulo isomorphisms) with precisely n unbounded edges will be denoted $\mathscr{M}_{0,n,\mathrm{trop}}$. (It can be thought of as a tropical analogue of the moduli space $\bar M_{0,n}$ of n-pointed stable rational curves.)

Remark 3.3. The isomorphism condition of definition 3.2 means that every edge of a marked tropical curve has a parametrization as an interval in $\mathbb{R}$ that is unique up to translations and sign. In particular, every bounded edge E of a tropical curve has an intrinsic *length* that we will denote by $l(E) \in \mathbb{R}_{>0}$.

One way to fix this translation and sign ambiguity is to pick a flag F of the edge E: there is then a unique choice of parametrization such that the corresponding closed interval is $[0,l(E)]$ (or $[0,\infty)$ for unbounded edges), with the chosen flag F being the zero point of this interval. We will call this the *canonical parametrization* of E with respect to the flag F.

Definition 3.4 (Combinatorial types)**.** The *combinatorial type* of a marked tropical curve $(\Gamma,x_1,\dots,x_n)$ is defined to be the homeomorphism class of Γ relative $x_1,\dots,x_n$ (i.e. the data of $(\Gamma,x_1,\dots,x_n)$ modulo homeomorphisms of Γ that map each x_i to itself).

Lemma 2.10 of [GM2] says that there are only finitely many combinatorial types of curves in $\mathscr{M}_{0,n}$. The subset of curves of type α in $\mathscr{M}_{0,n,\mathrm{trop}}$ is the interior of a cone in $\mathbb{R}^k$ (where k is the number of bounded edges) given by the inequalities that all lengths are positive — i.e. it is the positive orthant of $\mathbb{R}^k$. Example 2.13 of [GM2] describes how these cones are glued locally in $\mathscr{M}_{0,n,\mathrm{trop}}$.

Note that this construction is exactly the same as the construction of the *space of (phylogenetic) trees* $\mathbb{T}_n$ in [BHV], page 9–13.

Fix $n \geq 3$ and consider the space $\mathbb{R}^{\binom{n}{2}}$ indexed by the set $\mathscr{T}$ of all subsets $T \subset [n] := \{1,\dots,n\}$ with $|T| = 2$. In order to embed $\mathscr{M}_{0,n,\mathrm{trop}}$ into a quotient of $\mathbb{R}^{\binom{n}{2}}$, consider the map

$$\begin{array}{rcl} \mathrm{dist}_n : \mathscr{M}_{0,n,\mathrm{trop}} & \longrightarrow & \mathbb{R}^{\binom{n}{2}} \\ (\Gamma, x_1, \dots, x_n) & \longmapsto & (\mathrm{dist}_\Gamma(x_i, x_j))_{\{i,j\} \in \mathscr{T}} \end{array}$$

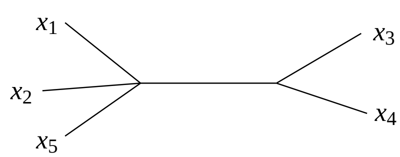
where $\mathrm{dist}_\Gamma(x_i, x_j)$ denotes the distance between the unbounded edges (or leaves) x_i and x_j, that is, the sum of the lengths of all edges on the (unique) path leading from x_i to x_j. Furthermore, define a linear map Φ_n by

$$\begin{array}{rcl} \Phi_n : \mathbb{R}^n & \longrightarrow & \mathbb{R}^{\binom{n}{2}} \\ a & \longmapsto & (a_i + a_j)_{\{i,j\} \in \mathscr{T}}. \end{array}$$

Denote by Q_n the $(\binom{n}{2} - n)$-dimensional quotient vector space $\mathbb{R}^{\binom{n}{2}} / \mathrm{Im}(\Phi_n)$, and by $q_n : \mathbb{R}^{\binom{n}{2}} \to Q_n$ the canonical projection.

Theorem 3.5 ([SS], theorem 4.2)**.** *The map $\varphi_n := q_n \circ \mathrm{dist}_n : \mathscr{M}_{0,n,\mathrm{trop}} \to Q_n$ is an embedding, and the image $\varphi_n(\mathscr{M}_{0,n,\mathrm{trop}}) \subset Q_n$ is a simplicial fan of pure dimension $n-3$. The interior of its k-dimensional cells corresponds to combinatorial types of graphs with n marked leaves and exactly k bounded edges.*

Construction 3.6. For each subset $I \subset [n]$ of cardinality $1 < |I| < n-1$, define v_I to be the image under φ_n of a tree with one bounded edge of length one, the marked ends with labels in I on one side of the bounded edge and the marked ends with labels in $[n] \setminus I$ on the other. Note that $v_I = v_{[n]\setminus I}$ by construction. As an example, the following picture shows the 5-marked rational curve corresponding to the vector $v_{\{1,2,5\}} = v_{\{3,4\}} \in Q_5$:



By theorem 3.5, the v_I generate the edges of the simplicial fan $\varphi_n(\mathscr{M}_{0,n,\mathrm{trop}})$. Let $\Lambda_n := \langle v_I \rangle_{\mathbb{Z}} \subset Q_n$ be the lattice in Q_n generated by the vectors v_I.

Theorem 3.7. *The marked fan $(\varphi_n(\mathscr{M}_{0,n,\mathrm{trop}}), \{v_I\})$ is a tropical fan in Q_n with lattice Λ_n. In other words, using the embedding φ_n the space $\mathscr{M}_{0,n,\mathrm{trop}}$ can be thought of as a tropical fan of dimension $n-3$.*

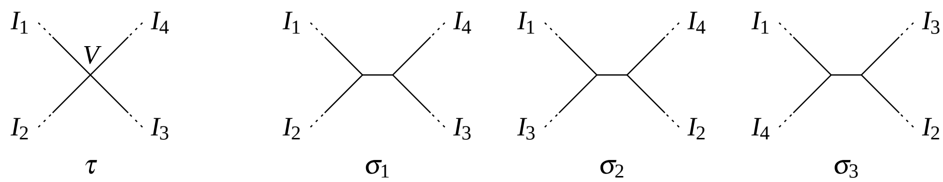
Proof. By lemma 2.14 we have to prove that

$$\sum_{\sigma > \tau} v_{\sigma/\tau} = 0 \in Q_n / V_\tau$$

for all $\tau \in \varphi_n(\mathscr{M}_{0,n,\mathrm{trop}})^{(n-4)}$ and $v_{\sigma/\tau}$ as in construction 2.13 (b).

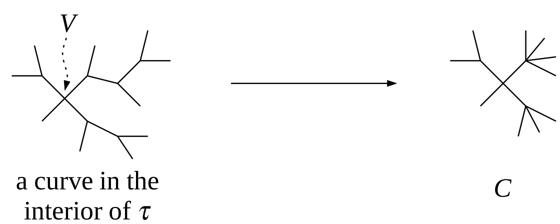
In order to prove this fix a cell $\tau \in \varphi_n(\mathscr{M}_{0,n,\mathrm{trop}})^{(n-4)}$. As τ is a cell of codimension one, the curves in the interior of τ have exactly one vertex V of valence 4 and all other vertices of valence 3. Denote by $I_s \subset [n]$ for $s = 1,\dots,4$ the set of marked ends lying behind the s-th edge adjacent to V. By construction there are exactly three combinatorial types $\sigma_1, \sigma_2, \sigma_3 > \tau$ in $\mathscr{M}_{0,n,\mathrm{trop}}$, obtained from τ by replacing the vertex V with an edge as in

the following picture (where we have only drawn the relevant part of the graphs and the labels denote the marked ends lying behind the corresponding edges):



Note that $v_{I_1 \cup I_2}$ is an edge of σ_1 which is not an edge of τ; hence we have $v_{\sigma_1/\tau} = v_{I_1 \cup I_2}$, and similarly $v_{\sigma_2/\tau} = v_{I_1 \cup I_3}$ and $v_{\sigma_3/\tau} = v_{I_1 \cup I_4}$.

Now let $C \in \tau$ be the point corresponding to the curve obtained from the combinatorial type τ by setting the lengths of each bounded edge E to 1 if E is adjacent to V, and 0 otherwise:



Note that this operation changes the combinatorial type in general. Furthermore, let $a \in \mathbb{R}^n$ be the vector with $a_i = 1$ if the marked end x_i is adjacent to V, and $a_i = 0$ otherwise. We claim that

$$v_{\sigma_1/\tau} + v_{\sigma_2/\tau} + v_{\sigma_3/\tau} = \Phi_n(a) + \mathrm{dist}_n(C) \in \mathbb{R}^{\binom{n}{2}},$$

from which the required balancing condition then follows after passing to the quotient by $\mathrm{Im}\,\Phi_n + V_\tau$. We check this equality coordinate-wise for all $T = \{i, j\} \in \mathscr{T}$. We may assume that i and j do not lie in the same I_s since otherwise the T-coordinate of every term in the equation is zero. Then the T-coordinate on the left hand side is 2 since x_i and x_j are on different sides of the newly inserted edge for exactly two of the types $\sigma_1, \sigma_2, \sigma_3$. On the other hand, if we denote by $0 \le c \le 2$ the number how many of the two ends x_i and x_j are adjacent to V, then the T-coordinates of $\Phi_n(a)$ and $\mathrm{dist}_n(C)$ are c and $2 - c$ respectively; so the proposition follows. □

Remark 3.8. A slightly different proof of this balancing condition has recently been found independently by Mikhalkin in [M3] section 2.

Remark 3.9. The space of trees appears in [SS] as a quotient of the tropical Grassmannian. Let us review how the tropical Grassmannian is defined. Tropical varieties can be defined as images of usual algebraic varieties over the field K of Puiseux series. An element of K is a Puiseux series $p(t) = a_1 t^{q_1} + a_2 t^{q_2} + a_3 t^{q_3} + \ldots$, where $a_i \in \mathbb{C}$ and $q_1 < q_2 < q_3 < \ldots$ are rational numbers which share a common denominator. Define the *valuation* of $p(t)$ to be $\mathrm{val}(p(t)) = -q_1$, if $p(t) \neq 0$, and $-\infty$ otherwise. Let $V(I)$ be a variety in $(K^*)^n$. Its *tropicalization* $\mathrm{Trop}(I)$ is defined to be the closure of the set

$$\{(\mathrm{val}(y_1), \ldots, \mathrm{val}(y_n)) \mid (y_1, \ldots, y_n) \in V(I)\} \subset \mathbb{R}^n.$$

The tropicalization of an ideal can also be computed using initial ideals: it is equal to the closure of the set of $w \in \mathbb{Q}^n$ such that the initial ideal $\text{in}_w I$ contains no monomial (see theorem 2.1 of [SS]). In this way, it can be considered as a subfan of the Gröbner fan of I and inherits its fan structure (this is explained for example in [BJSST]). Theorem 2.5.1 of [S] shows that $\text{Trop}(I)$ is a tropical fan.

In [SS], the *tropical Grassmannian* $\mathscr{G}_{2,n}$ is defined to be the tropicalization of the ideal $I_{2,n}$ of the usual Grassmannian in its Plücker embedding. The ideal $I_{2,n}$ is an ideal in the polynomial ring with $\binom{n}{2}$ variables. For example, we have

$$I_{2,4} = \langle p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} \rangle \subset K[p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}].$$

(We label the variables with subsets of size 2 of $\{1,\dots,n\}$.) The tropical Grassmannian $\mathscr{G}_{2,n} := \text{Trop}(I_{2,n})$ is a tropical fan in $\mathbb{R}^{\binom{n}{2}}$.

The lineality space of $\mathscr{G}_{2,n}$ (that is, the intersection of all cones of $\mathscr{G}_{2,n}$) is equal to the image of Φ_n (see [SS], page 6). As $\text{Im}(\Phi_n)$ is contained in every cone of $\mathscr{G}_{2,n}$, we can reduce $\mathscr{G}_{2,n}$ modulo $\text{Im}(\Phi_n)$ and define the *reduced tropical Grassmannian*

$$\mathscr{G}'_{2,n} := \mathscr{G}_{2,n} / \text{Im}(\Phi_n) \subset \mathbb{R}^{\binom{n}{2}} / \text{Im}(\Phi_n).$$

Then $\mathscr{G}'_{2,n}$ is a tropical fan, too. Theorem 3.4 of [SS] states that $\mathscr{G}'_{2,n}$ is equal to the space of phylogenetic trees $\mathbb{T}_n$.

Example 3.10. For $n = 4$ the space $\mathscr{M}_{0,4,\text{trop}}$ is embedded by φ_4 in $Q_4 = \mathbb{R}^{\binom{4}{2}} / \text{Im}\,\Phi_n \cong \mathbb{R}^6/\mathbb{R}^4 \cong \mathbb{R}^2$, and the lattice $\Lambda_4 \subset Q_4$ is spanned by the three vectors $v_{\{1,2\}} = v_{\{3,4\}}$, $v_{\{1,3\}} = v_{\{2,4\}}$, and $v_{\{1,4\}} = v_{\{2,3\}}$. The space $\mathscr{M}_{0,4,\text{trop}}$ is of dimension 1 and has three maximal cones, each spanned by one of the three vectors above. By the balancing condition of theorem 3.7 the sum of these three vectors is 0 in Q_4, and hence any two of them form a basis of Λ_4. With (the negative of) such a basis $\varphi_4(\mathscr{M}_{0,4,\text{trop}})$ simply becomes the standard tropical line $L^2_1 \subset \mathbb{R}^2$ as in example 2.5 (c).

For the rest of this chapter we will now consider the forgetful maps between the moduli spaces of abstract curves and show that they are morphisms of fans. For simplicity we will only consider the map that forgets the last marked end.

Definition 3.11 (Forgetful maps)**.** Let $n \geq 4$ be an integer. We have a *forgetful map* ft from $\mathscr{M}_{0,n,\text{trop}}$ to $\mathscr{M}_{0,n-1,\text{trop}}$ which assigns to an n-marked curve $(C, x_1, \dots, x_n)$ the (stabilization of the) $(n-1)$-marked curve $(C, x_1, \dots, x_{n-1})$ (see [GM2], definition 4.1). By stabilization we mean that we have to straighten 2-valent vertices that possibly emerge when removing the marked end x_n.

Proposition 3.12. *With the tropical fan structure of theorem 3.7 the forgetful map* $\text{ft} : \mathscr{M}_{0,n,\text{trop}} \to \mathscr{M}_{0,n-1,\text{trop}}$ *is a morphism of fans in the sense of definition 2.22.*

Proof. Let $\text{pr} : \mathbb{R}^{\binom{n}{2}} \to \mathbb{R}^{\binom{n-1}{2}}$ denote the projection to those coordinates $T \in \mathscr{T}$ with $n \notin T$. As $\text{pr}(\text{Im}(\Phi_n)) = \text{Im}(\Phi_{n-1})$, the map pr induces a linear map $\widetilde{\text{pr}} : Q_n \to Q_{n-1}$.

We claim that $\widetilde{\mathrm{pr}}_{|\varphi_n(M_{0,n,\mathrm{trop}})}$ is the map induced by ft, hence we have to show the commutativity of the diagram

$$\begin{tikzcd}
\mathscr{M}_{0,n,\mathrm{trop}} \arrow[r, "\mathrm{ft}"] \arrow[d, "\varphi_n"'] & \mathscr{M}_{0,n-1,\mathrm{trop}} \arrow[d, "\varphi_{n-1}"] \\
Q_n \arrow[r, "\widetilde{\mathrm{pr}}"] & Q_{n-1}.
\end{tikzcd}$$

So let $C = (\Gamma, x_1, \ldots, x_n) \in \mathscr{M}_{0,n,\mathrm{trop}}$ be an abstract n-marked rational tropical curve. If x_n is not adjacent to exactly one bounded and one unbounded edge, then forgetting x_n does not change any of the distances $\mathrm{dist}_C(x_i, x_j)$ for $i, j \neq n$; so in this case we are done.

Now assume that x_n is adjacent to one unbounded edge x_k and one bounded edge E of length $l(E)$. Then the distances between marked points are given by

$$\mathrm{dist}_{\mathrm{ft}(C)}(x_i, x_j) = \begin{cases} \mathrm{dist}_C(x_i, x_j) - l(E) & \text{if } j = k \\ \mathrm{dist}_C(x_i, x_j) & \text{if } j \neq k, \end{cases}$$

hence $\widetilde{\mathrm{pr}}(\varphi_n(C)) = \varphi_{n-1}(\mathrm{ft}(C)) + \Phi_{n-1}(l(E) \cdot e_k)$, where e_k denotes the k-th standard unit vector in $\mathbb{R}^{n-1}$. We conclude that $\widetilde{\mathrm{pr}} \circ \varphi_n = \varphi_{n-1} \circ \mathrm{ft}$. Hence the diagram is commutative.

It remains to check that $\widetilde{\mathrm{pr}}(\Lambda_n) \subset \Lambda_{n-1}$. It suffices to check this on the generators v_I; and we can assume $n \in I$ since $v_I = v_{[n]\setminus I}$. We get

$$\widetilde{\mathrm{pr}}(v_I) = \begin{cases} v_{I\setminus\{n\}} & , \text{if } |I| \geq 3 \\ 0 & , \text{if } |I| = 2. \end{cases}$$

□

4. Tropical curves in $\mathbb{R}^r$

In this chapter we will consider tropical analogues of the algebro-geometric moduli spaces of (rational) stable maps as e.g. already introduced in [GM2]. Similarly to the classical case the points of these spaces correspond to marked abstract tropical curves together with a suitable map to some $\mathbb{R}^r$. We would like to make these spaces into tropical fans by considering the underlying abstract tropical curves and using the fan structure of $\mathscr{M}_{0,n,\mathrm{trop}}$ developed in chapter 3. In order for this to work it turns out however that we have to modify the construction of [GM2] slightly: in the underlying abstract tropical curves we have to label *all* ends and not just the contracted ones corresponding to the marked points.

Definition 4.1 (Tropical $\mathscr{M}^{\mathrm{lab}}_{0,n,\mathrm{trop}}(\mathbb{R}^r, \Delta)$)**.** A *(parametrized) labeled n-marked tropical curve in* $\mathbb{R}^r$ is a tuple $(\Gamma, x_1, \ldots, x_N, h)$ for some $N \geq n$, where $(\Gamma, x_1, \ldots, x_N)$ is an abstract N-marked tropical curve with exactly N ends and $h : \Gamma \to \mathbb{R}^r$ is a continuous map satisfying:

(a) On each edge of Γ the map h is of the form $h(t) = a + t \cdot v$ for some $a \in \mathbb{R}^r$ and $v \in \mathbb{Z}^r$. The integral vector v occurring in this equation if we pick for E the canonical parametrization starting at $V \in \partial E$ (see 3.3) will be denoted $v(E, V)$ and called the *direction* of E (at V). If E is an unbounded edge and V its only boundary point we will write for simplicity $v(E)$ instead of $v(E, V)$.

(b) For every vertex V of Γ we have the *balancing condition*

$$\sum_{E \mid V \in \partial E} v(E, V) = 0.$$

(c) $v(x_i) = 0$ for $i = 1,\dots,n$ (i.e. each of the first n ends is contracted by h), whereas $v(x_i) \neq 0$ for $i > n$ (i.e. the remaining $N-n$ ends are "non-contracted ends").

Two labeled n-marked tropical curves $(\Gamma, x_1,\dots,x_N,h)$ and $(\tilde{\Gamma},\tilde{x}_1,\dots,\tilde{x}_N,\tilde{h})$ in $\mathbb{R}^r$ are called isomorphic (and will from now on be identified) if there is an isomorphism $\varphi : (\Gamma,x_1,\dots,x_N) \to (\tilde{\Gamma},\tilde{x}_1,\dots,\tilde{x}_N)$ of the underlying abstract curves such that $\tilde{h}\circ\varphi = h$.

The *degree* of a labeled n-marked tropical curve as above is defined to be the $(N-n)$-tuple $\Delta = (v(x_{n+1}),\dots,v(x_N)) \in (\mathbb{Z}^r\backslash\{0\})^{N-n}$ of directions of its non-contracted ends. Its *combinatorial type* is given by the data of the combinatorial type of the underlying abstract marked tropical curve $(\Gamma,x_1,\dots,x_N)$ together with the directions of all its (bounded as well as unbounded) edges. For the rest of this work the number N will always be related to n and Δ by $N = n + \#\Delta$ and thus denote the total number of (contracted or non-contracted) ends of an n-marked curve in $\mathbb{R}^r$ of degree Δ.

The space of all labeled n-marked tropical curves of a given degree Δ in $\mathbb{R}^r$ will be denoted $\mathscr{M}^{\text{lab}}_{0,n,\text{trop}}(\mathbb{R}^r,\Delta)$. For the special choice

$$\Delta = (-e_0,\dots,-e_0 \;,\dots,\; -e_r,\dots,-e_r)$$

with $e_0 := -e_1 - \cdots - e_r$ and where each e_i occurs exactly d times we will also denote this space by $\mathscr{M}^{\text{lab}}_{0,n,\text{trop}}(\mathbb{R}^r,d)$ and say that these curves have degree d.

Definition 4.2 (Evaluation map)**.** For $i = 1,\dots,n$ the map

$$\begin{aligned} \text{ev}_i : \mathscr{M}^{\text{lab}}_{0,n,\text{trop}}(\mathbb{R}^r,\Delta) &\quad\to\quad \mathbb{R}^r \\ (\Gamma,x_1,\dots x_N,h) &\quad\longmapsto\quad h(x_i) \end{aligned}$$

is called the *i-th evaluation map* (note that this is well-defined for the contracted ends since for them $h(x_i)$ is a point in $\mathbb{R}^r$).

Construction 4.3 (Tropical $\mathscr{M}_{0,n,\text{trop}}(\mathbb{R}^r,\Delta)$). Let $N \geq n \geq 0$, and let $\Delta = (v_{n+1},\dots,v_N) \in (\mathbb{Z}^r\backslash\{0\})^{N-n}$. It is obvious from the definitions that the subgroup G of the symmetric group $\mathbb{S}_{N-n}$ consisting of all permutations σ of $\{n+1,\dots,N\}$ such that $v_{\sigma(i)} = v_i$ for all $i = n+1,\dots,N$ acts on the space $\mathscr{M}^{\text{lab}}_{0,n,\text{trop}}(\mathbb{R}^r,\Delta)$ by relabeling the non-contracted ends. We denote the quotient space $\mathscr{M}^{\text{lab}}_{0,n,\text{trop}}(\mathbb{R}^r,\Delta)/G$ by $\mathscr{M}_{0,n,\text{trop}}(\mathbb{R}^r,\Delta)$ and call this the space of *(unlabeled) n-marked tropical curves* in $\mathbb{R}^r$ of degree Δ in $\mathbb{R}^r$; its elements can obviously be thought of as n-marked tropical curves in $\mathbb{R}^r$ for which we have only specified how many of its ends have a given direction, but where no labeling of these (non-contracted) ends is given. Consequently, when considering unlabeled curves we can (and usually will) think of Δ as a multiset $\{v_{n+1},\dots,v_N\}$ instead of as a vector $(v_{n+1},\dots,v_N)$.

Note that this definition of $\mathscr{M}_{0,n,\text{trop}}(\mathbb{R}^r,\Delta)$ agrees with the one given in [GM2], and that for curves of degree d the group G above is precisely $(\mathbb{S}_d)^{r+1}$.

Remark 4.4. Proposition 2.1 of [NS] tells us that there are only finitely many combinatorial types in any given moduli space $\mathscr{M}_{0,n,\text{trop}}(\mathbb{R}^r,\Delta)$ (and thus also in $\mathscr{M}^{\text{lab}}_{0,n,\text{trop}}(\mathbb{R}^r,\Delta)$). Proposition 2.11 of [GM2] says that the subset of curves of a fixed combinatorial type is a cone in a real vector space, and example 2.13 shows how these cones are glued locally. (These results are only stated there for plane tropical curves, but it is obvious that they hold in the same way if we replace $\mathbb{R}^2$ by $\mathbb{R}^r$.) Hence the moduli spaces of labeled or unlabeled n-marked tropical curves in $\mathbb{R}^r$ are polyhedral complexes.

Remark 4.5. We will now show how one can use the tropical fan structure of $\mathscr{M}_{0,n,\text{trop}}$ to make the moduli spaces $\mathscr{M}^{\text{lab}}_{0,n,\text{trop}}(\mathbb{R}^r,d)$ of labeled curves in $\mathbb{R}^r$ into tropical fans as well. The idea is the same as in the classical algebro-geometric case: the non-contracted ends of the tropical curves in $\mathbb{R}^r$ can be thought of as their intersection points with the $r+1$ coordinate hyperplanes of $\mathbb{P}^r$; so passing from unlabeled to labeled curves corresponds to labeling these intersection points (which is one possible strategy to construct the moduli spaces of stable maps in algebraic geometry). The final moduli spaces of stable maps then arise by taking the quotient by the group of possible permutations of the labels. Note that this makes the spaces of stable maps into stacks instead of varieties (as the group action is in general not free); and similarly our moduli spaces of unlabeled tropical curves in $\mathbb{R}^r$ can only be thought of as "tropical stacks" instead of tropical varieties. To avoid this notion of tropical stacks (which has not been developed yet) we will always work with labeled tropical curves in this paper. Of course this is not a problem for enumerative purposes since we can always count labeled curves first and then divide the result by $|G|$ in the end.

Now let us make the relation between the spaces $\mathscr{M}^{\text{lab}}_{0,n,\text{trop}}(\mathbb{R}^r,\Delta)$ and $\mathscr{M}_{0,N,\text{trop}}$ (with $N = n+\#\Delta$) precise. For this we consider the forgetful map

$$\begin{aligned}\psi: \quad \mathscr{M}^{\text{lab}}_{0,n,\text{trop}}(\mathbb{R}^r,\Delta) &\to \mathscr{M}_{0,n+\#\Delta,\text{trop}}\\ (\Gamma,x_1,\dots,x_N,h) &\mapsto (\Gamma,x_1,\dots,x_N)\end{aligned}$$

that forgets the map to $\mathbb{R}^r$ but keeps all (contracted and non-contracted) unbounded edges. For this construction (and the rest of this paper) we will assume for simplicity that $N \geq 3$, so that this map is well-defined.

Lemma 4.6. *The forgetful map ψ induces a bijection of combinatorial types of the two moduli spaces $\mathscr{M}^{\text{lab}}_{0,n,\text{trop}}(\mathbb{R}^r,\Delta)$ and $\mathscr{M}_{0,N,\text{trop}}$ with $N = n+\#\Delta$.*

Proof. It is clear that ψ induces a well-defined map between combinatorial types of labeled and abstract tropical curves. By definition the only additional data in the combinatorial type of a labeled curve compared to the underlying abstract curve is the directions of the edges. As the directions of the ends are fixed by Δ it therefore suffices to show that for a given (combinatorial type of a) graph Γ and fixed directions of the ends there is a unique choice of directions for the bounded edges compatible with the balancing condition of definition 4.1.

We will prove this by induction on N. The statement is clear for $N = 3$ since there are no bounded edges in this case. In order to prove the induction step we show first that there is a vertex V_1 of valence l with $l-1$ ends adjacent. Assume there is no such vertex. Then

$$N \leq \sum_V (\operatorname{val} V - 2) = \sum_V \operatorname{val} V - 2\cdot k,$$

where k denotes the number of vertices. As

$$k = N-2-\sum_V(\operatorname{val} V-3) = N-2-\sum_V \operatorname{val} V + 3k$$

we have

$$N \leq \sum_V \operatorname{val} V - \sum_V \operatorname{val} V + N - 2$$

which is a contradiction.

So assume that the (contracted or non-contracted) ends $x_{i_1},\dots,x_{i_{l-1}}$ are adjacent at a vertex V_1 of valence l. Let E be the only bounded edge which is adjacent to V_1. Then the directions

of $x_{i_1},\dots,x_{i_{l-1}}$ are fixed (by Δ for the non-contracted ends, and to be 0 for the contracted ones), so there is a unique choice for the direction of E compatible with the balancing condition at V_1. Now remove the unbounded edges $x_{i_1},\dots,x_{i_{l-1}}$ from Γ and make E into an unbounded edge with this new fixed direction. This new graph has $N-(l-1)+1$ ends, which is less than N as l is at least 3. Therefore we can apply induction and conclude that the directions of all bounded edges are determined. $\square$

For the rest of this paper we will assume for simplicity that $n>0$, i.e. that there is at least one contracted end.

Proposition 4.7. *With notations as above, the map*

$$\begin{aligned}\mathscr{M}^{\text{lab}}_{0,n,\text{trop}}(\mathbb{R}^r,\Delta) &\to Q_N\times\mathbb{R}^r\\ C &\mapsto (\varphi_N(\psi(C)),\text{ev}_1(C))\end{aligned}$$

is an embedding whose image is the tropical fan $\varphi_N(\mathscr{M}_{0,N,\text{trop}})\times\mathbb{R}^r$. So $\mathscr{M}^{\text{lab}}_{0,n,\text{trop}}(\mathbb{R}^r,\Delta)$ can be thought of as a tropical fan of dimension $r+N-3=r+n+\#\Delta-3$, namely as the fan $\mathscr{M}_{0,n+\#\Delta,\text{trop}}\times\mathbb{R}^r$.

Proof. It is clear that the given map is a continuous map of polyhedral complexes. By lemma 4.6 it suffices to check injectivity and compute its image for a fixed combinatorial type α. As in proposition 2.11 of [GM2] the cell of $\mathscr{M}^{\text{lab}}_{0,n,\text{trop}}(\mathbb{R}^r,\Delta)$ corresponding to curves of type α is given by $\mathbb{R}^k_{>0}\times\mathbb{R}^r$, where the first positive coordinates are the lengths of the bounded edges, and the second r coordinates are the position of a root vertex (that we choose to be the vertex V adjacent to x_1 here). This is true because we can recover the map h if we know the position $h(V)$ and the lengths of each bounded edge (in addition to the information of α). The map of the proposition obviously sends this cell bijectively to the product of the corresponding cell of $\varphi_N(\mathscr{M}_{0,N,\text{trop}})$ and $\mathbb{R}^r$ (since the position of the root vertex $h(V)$ is unrestricted). $\square$

Proposition 4.8. *With the tropical fan structure of proposition 4.7 the evaluation maps $\text{ev}_i:\mathscr{M}^{\text{lab}}_{0,n,\text{trop}}(\mathbb{R}^r,\Delta)\to\mathbb{R}^r$ are morphisms of fans (in the sense of definition 2.22).*

Proof. As usual let $N=n+\#\Delta$, and identify $\mathscr{M}^{\text{lab}}_{0,n,\text{trop}}(\mathbb{R}^r,\Delta)$ with the space $\mathscr{M}_{0,N,\text{trop}}\times\mathbb{R}^r$ as in proposition 4.7. For all $1\le i\le n$ consider the linear map

$$\begin{aligned}\text{ev}'_i:\mathbb{R}^{\binom{N}{2}}\times\mathbb{R}^r &\longrightarrow \mathbb{R}^r\\ (a_{\{1,2\}},\dots,a_{\{N-1,N\}},b) &\longmapsto b+\frac{1}{2}\sum_{\substack{k=2\\k\neq i}}^{N}\left(a_{\{1,k\}}-a_{\{i,k\}}\right)v_k,\end{aligned}$$

where v_k is the direction of the k-th end, i.e. $v_1=\cdots=v_n=0$ and $(v_{n+1},\dots,v_N)=\Delta$. As $\text{ev}'_i(\text{Im}(\Phi_N)\times\{0\})=\{0\}$, the map ev'_i induces a linear map $\widetilde{\text{ev}'_i}:Q_N\times\mathbb{R}^r\to\mathbb{R}^r$.

We claim that $\widetilde{\text{ev}'_i}|_{\mathscr{M}_{0,N,\text{trop}}\times\mathbb{R}^r}$ is the map induced by ev_i, hence we have to show the commutativity of the diagram

$$\begin{tikzcd}\mathscr{M}_{0,N,\text{trop}}\times\mathbb{R}^r \arrow[r,"\text{ev}_i"] \arrow[d,"\varphi_N\times\text{id}"'] & \mathbb{R}^r\\ Q_N\times\mathbb{R}^r \arrow[ur,"\widetilde{\text{ev}'_i}"'] & \end{tikzcd}$$

As the case $i=1$ is trivial, and using the additivity of the function $\sum_{k=1}^N v_k\,(a_{1k}-a_{ik})$, we may assume that there exists only one edge E, and that this edge separates x_1 and x_i. Let V be the vertex adjacent to x_1, and let $T_1 \subset [N]$ be the set of ends lying on the connected component of $\Gamma \setminus \{E\}$ containing x_1 (so that $1 \in T$ and $i \notin T$). Since by the balancing condition of definition 4.1 the equation $v(E,V) = \sum_{k\notin T_1} v_k = -\sum_{k\in T_1} v_k$ holds, we get

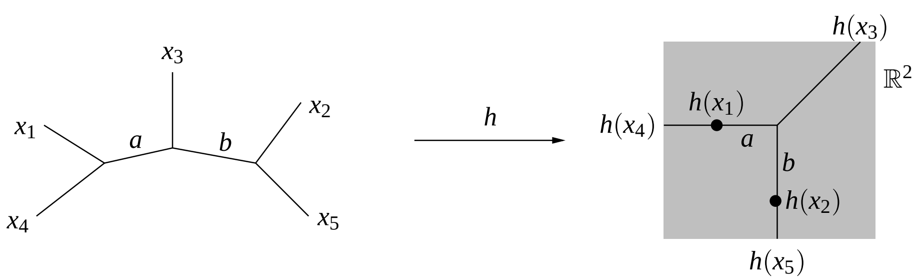
$$\begin{aligned}\left(\widetilde{\mathrm{ev}}'_i \circ (\varphi_N \times id)\right)(\Gamma, x_1, \ldots, x_N, h(x_1)) &= h(x_1) + \frac{1}{2}\left[-\sum_{k\in T_1} l(E)v_k + \sum_{k\notin T_1} l(E)v_k\right] \\ &= h(x_1) + l(E)\,v(E,V) \\ &= h(x_i)\end{aligned}$$

as required (where $l(E)$ denotes the length of E). It remains to check that $\widetilde{\mathrm{ev}}'_i(\Lambda_N \times \mathbb{Z}^r) \subset \mathbb{Z}^r$. It suffices to check that $\widetilde{\mathrm{ev}}'_i(v_I, b) \in \mathbb{Z}^r$ for the generators v_I of Λ_N (see construction 3.6) and all $b \in \mathbb{Z}^r$. Assuming $1 \in I$, we get

$$\widetilde{\mathrm{ev}}'_i(v_I, b) = \begin{cases} b - \sum_{k\in I} v_k & \text{, if } i \notin I \\ b & \text{, if } i \in I, \end{cases}$$

which finishes the proof as $v_k \in \mathbb{Z}^r$. □

Example 4.9. As an example of the calculation in the above proof we consider the space $\mathscr{M}^{\text{lab}}_{0,2,\text{trop}}(\mathbb{R}^2, 1) = \mathscr{M}_{0,5,\text{trop}} \times \mathbb{R}^2$, so that the curves have $N = 5$ unbounded edges with directions $v_1 = v_2 = 0$, $v_3 = (1,1)$, $v_4 = (-1,0)$, $v_5 = (0,-1)$. We consider curves of the combinatorial type drawn in the following picture:



We can read off immediately from the picture that $\mathrm{ev}_2(C) = h(x_1) + a(1,0) + b(0,-1)$.

On the other hand, we compute using the formula of the proof of proposition 4.8:

$$\begin{aligned}&\left(\widetilde{\mathrm{ev}}'_2 \circ (\varphi_N \times id)\right)(C, x_1, \ldots, x_5, h(x_1)) \\ &= \widetilde{\mathrm{ev}}'_2(\mathrm{dist}_C(x_1,x_2), \ldots, \mathrm{dist}_C(x_4,x_5), h(x_1)) \\ &= h(x_1) + \frac{1}{2}\sum_{k=3}^{5} (\mathrm{dist}_C(x_1,x_k) - \mathrm{dist}_C(x_i,x_k))\,v_k \\ &= h(x_1) + \frac{1}{2}\left[(a-b)v_3 + (0-(a+b))v_4 + ((a+b)-0)v_5\right] \\ &= h(x_1) + \frac{1}{2}\left[a(v_3 - v_4 + v_5) + b(-v_3 - v_4 + v_5)\right] \\ &= h(x_1) + a(1,0) + b(0,-1).\end{aligned}$$

Remark 4.10. Just as in chapter 3 we can define forgetful maps on the moduli space of labeled tropical curves in $\mathbb{R}^r$ as well. We can forget certain contracted ends, but also we can forget all non-contracted ends and the $\mathbb{R}^r$ factor in $\mathscr{M}^{\text{lab}}_{0,n,\text{trop}}(\mathbb{R}^r,\Delta) = \mathscr{M}_{0,N,\text{trop}} \times \mathbb{R}^r$, which corresponds to forgetting the map h. The same argument as in proposition 3.12 shows that these forgetful maps are morphisms as well.

Remark 4.11. In proposition 4.7 we have constructed the tropical fan structure on the moduli space $\mathscr{M}^{\text{lab}}_{0,n,\text{trop}}(\mathbb{R}^r,\Delta)$ using the evaluation at the first (contracted) marked end. If we use a different contracted end x_i with $i \in \{2,\dots,n\}$ instead, the two embeddings in $Q_N \times \mathbb{R}^r$ only differ by addition of $\text{ev}_i(C) - \text{ev}_1(C)$ in the $\mathbb{R}^r$ factor. As this is an isomorphism by proposition 4.8 it follows that the tropical fan structure defined in proposition 4.7 is natural in the sense that it does not depend on the choice of contracted end.

5. Applications

In this final chapter we want to apply our results to reprove and generalize two statements in tropical enumerative geometry that have occurred earlier in the literature. The first application simply concerns the number of rational tropical curves in some $\mathbb{R}^r$ through given conditions.

Theorem 5.1. *Let $r \geq 2$, let Δ be a degree of tropical curves in $\mathbb{R}^r$ (i.e. a multiset of elements of $\mathbb{Z}^r \backslash \{0\}$ that sum up to 0), and let $n > 0$ be such that $r + n + \#\Delta - 3 = nr$. Then the number of rational tropical curves of degree Δ in $\mathbb{R}^r$ through n points in general position (counted with multiplicities) does not depend on the position of the points.*

Proof. Let $\text{ev} := \text{ev}_1 \times \cdots \times \text{ev}_n : \mathscr{M}^{\text{lab}}_{0,n,\text{trop}}(\mathbb{R}^r,\Delta) \to \mathbb{R}^{nr}$. By proposition 4.7 the space $\mathscr{M}^{\text{lab}}_{0,n,\text{trop}}(\mathbb{R}^r,\Delta)$ is a tropical fan, and by proposition 4.8 the map ev is a morphism of fans (of the same dimension). As $\mathbb{R}^{nr}$ is irreducible we know from corollary 2.26 that for general $Q \in \mathbb{R}^{nr}$ the number

$$\sum_{C \in |\mathscr{M}^{\text{lab}}_{0,n,\text{trop}}(\mathbb{R}^r,\Delta)| : \text{ev}(C) = Q} \text{mult}_C \, \text{ev}$$

does not depend on Q. So if we define the tropical multiplicity of a curve C occurring in this sum to be $\text{mult}_C \, \text{ev}$ we conclude that the number of labeled curves of degree Δ in $\mathbb{R}^r$ through the n points in $\mathbb{R}^r$ specified by Q does not depend on the choice of (general) Q. The statement for unlabeled curves now follows simply by dividing the resulting number by the order $|G|$ of the symmetry group of the non-contracted ends as in construction 4.3. □

Remark 5.2. In the proof of theorem 5.1 we have defined the tropical multiplicity of a curve to be the multiplicity of the evaluation map as in corollary 2.26. By corollary 2.27 this multiplicity can be computed on a fixed cell of $\mathscr{M}^{\text{lab}}_{0,n,\text{trop}}(\mathbb{R}^r,\Delta)$ as the absolute value of the determinant of the matrix obtained by expressing the evaluation map in terms of the basis on the source given by the vectors v_I (see construction 3.6) that span the given cell. But note that the coordinates on the given cell with respect to this basis are simply the lengths of the bounded edges. It follows that the determinant that we have to consider is precisely the same as that used in [GM2] chapter 3. In particular, in the case $r = 2$ it follows by proposition 3.8 of [GM2] that our multiplicity of the curves agrees with Mikhalkin's usual notion of multiplicity as in definitions 4.15 and 4.16 of [M1].

Remark 5.3. For simplicity we have formulated theorem 5.1 only for the case of counting tropical curves through given points. Of course the very same proof can be used for counting curves through given affine linear subspaces (with rational slopes) if one replaces the evaluation maps ev_i by their compositions with the quotient maps $\mathbb{R}^r \to \mathbb{R}^r/L_i$, where L_i is the linear subspace chosen at the i-th contracted end. This setup has been considered e.g. by Nishinou and Siebert in [NS].

As our second application we consider the map occurring in the proof of the tropical Kontsevich formula (see proposition 4.4 of [GM2]).

Theorem 5.4. *Let $d \geq 1$, and let $n = 3d$. Define*

$$\pi := \mathrm{ev}_1^1 \times \mathrm{ev}_2^2 \times \mathrm{ev}_3 \times \cdots \times \mathrm{ev}_n \times \mathrm{ft}_4 : \mathscr{M}_{0,n,\mathrm{trop}}(\mathbb{R}^2, d) \to \mathbb{R}^{2n-2} \times \mathscr{M}_{0,4,\mathrm{trop}},$$

i.e. π describes the first coordinate of the first marked point, the second coordinate of the second marked point, both coordinates of the other marked points, and the point in $\mathscr{M}_{0,4,\mathrm{trop}}$ defined by the first four marked points. Then $\deg_\pi(Q) := \sum_{P\in\pi^{-1}(Q)} \mathrm{mult}_\pi(P)$ (where $\mathrm{mult}_\pi(P)$ is defined in [GM2]*, definition 3.1) does not depend on Q (as long as Q is in general position).*

Proof. We define the map

$$\pi' := \mathrm{ev}_1^1 \times \mathrm{ev}_2^2 \times \mathrm{ev}_3 \times \cdots \times \mathrm{ev}_n \times \mathrm{ft}_4 : \mathscr{M}^{\mathrm{lab}}_{0,n,\mathrm{trop}}(\mathbb{R}^2, d) \to \mathbb{R}^{2n-2} \times \mathscr{M}_{0,4,\mathrm{trop}}$$

obtained from π by replacing $\mathscr{M}_{0,n,\mathrm{trop}}(\mathbb{R}^2,d)$ by $\mathscr{M}^{\mathrm{lab}}_{0,n,\mathrm{trop}}(\mathbb{R}^2,d)$. Then for each inverse image $P \in \mathscr{M}_{0,n,\mathrm{trop}}(\mathbb{R}^2,d)$ with $\pi(P) = Q$ there exist $|\mathbb{S}_d^3|$ different $P' \in \mathscr{M}^{\mathrm{lab}}_{0,n,\mathrm{trop}}(\mathbb{R}^2,d)$ with $\pi'(P') = Q$ of the same multiplicity $\mathrm{mult}_{\pi'}(P') = \mathrm{mult}_\pi(P)$ (for the different labelings of the non-marked unbounded edges). Hence

$$|\mathbb{S}_d^3| \cdot \deg_\pi(Q) = \deg_{\pi'}(Q).$$

So it is enough to show that $\deg_{\pi'}(Q)$ does not depend on Q. By proposition 4.7 the space $\mathscr{M}^{\mathrm{lab}}_{0,n,\mathrm{trop}}(\mathbb{R}^2,d)$ is a tropical fan. By example 3.10 the space $\mathscr{M}_{0,4,\mathrm{trop}}$ is just a tropical line in $\mathbb{R}^2$ and thus an irreducible tropical fan by example 2.18. As $\mathbb{R}^{2n-2}$ is an irreducible tropical fan (consisting of just one cone), too, we can conclude using proposition 2.20 that $Y := \mathbb{R}^{2n-2} \times \mathscr{M}_{0,4,\mathrm{trop}}$ is an irreducible tropical fan. Obviously, the source and target of π' are of the same dimension. As π' is a morphism of fans by propositions 3.12 and 4.8 we can apply corollary 2.26 and conclude that

$$\sum_{P \in |\mathscr{M}^{\mathrm{lab}}_{0,n,\mathrm{trop}}(\mathbb{R}^2,d)| : \pi'(P)=Q} \mathrm{mult}_P\, \pi'$$

does not depend on Q. So it remains to show that $\mathrm{mult}_P\, \pi' = \mathrm{mult}_{\pi'}(P)$. But this follows from corollary 2.27 in the same way as in remark 5.2. □

Remark 5.5. In the same way as in theorem 5.1 the result of theorem 5.4 can of course be generalized immediately to the case of curves in $\mathbb{R}^r$ of arbitrary degree Δ and with various linear subspaces as conditions.

REFERENCES

[BHV] L. Billera, S. Holmes, K. Vogtmann, *Geometry of the space of phylogenetic trees*, Adv. in Appl. Math. **27** (2001), 733–767.

[BJSST] T. Bogart, A. Jensen, D. Speyer, B. Sturmfels, R. Thomas, *Computing tropical varieties*, J. Symbolic Comput. **42** (2007), 54–73.

[GM1] A. Gathmann, H. Markwig, *The Caporaso-Harris formula and plane relative Gromov-Witten invariants in tropical geometry*, Math. Ann. **338** (2007), 845–868.

[GM2] A. Gathmann, H. Markwig, *Kontsevich's formula and the WDVV equations in tropical geometry*, Adv. Math. **217** (2008), 537–560.

[M1] G. Mikhalkin, *Enumerative tropical geometry in* $\mathbb{R}^2$, J. Amer. Math. Soc. **18** (2005), 313–377.

[M2] G. Mikhalkin, *Tropical Geometry and its applications*, Proceedings of the ICM, Madrid, Spain (2006), 827–852, preprint math.AG/0601041.

[M3] G. Mikhalkin, *Moduli spaces of rational tropical curves*, preprint arXiv/0704.0839.

[NS] T. Nishinou, B. Siebert, *Toric degenerations of toric varieties and tropical curves*, Duke Math. J. **135** (2006), 1–51.

[S] D. Speyer, *Tropical Geometry*, PhD thesis, University of California, Berkeley (2005).

[SS] D. Speyer, B. Sturmfels, *The tropical Grassmannian*, Adv. Geom. **4** (2004), 389–411.

[ST] B. Sturmfels, J. Tevelev, *Elimination theory for tropical varieties*, Math. Res. Lett. (to appear).

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